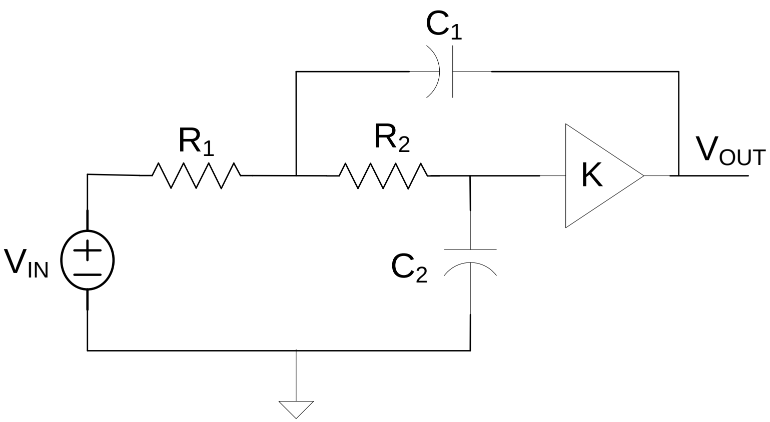
EE 508
Lecture 20

Sensitivity Functions

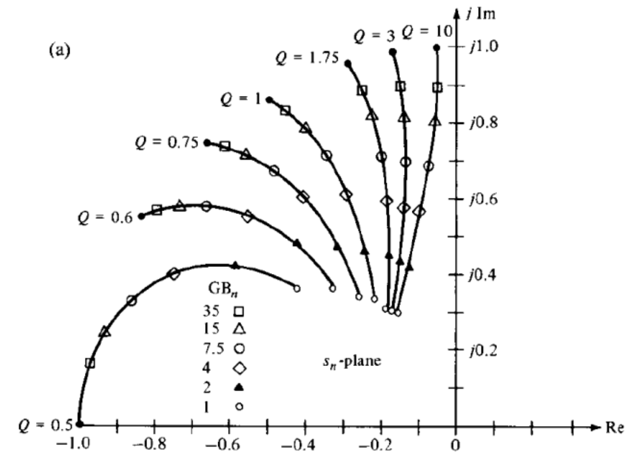
- Comparison of Filter Structures
- Performance Prediction

Review from last time

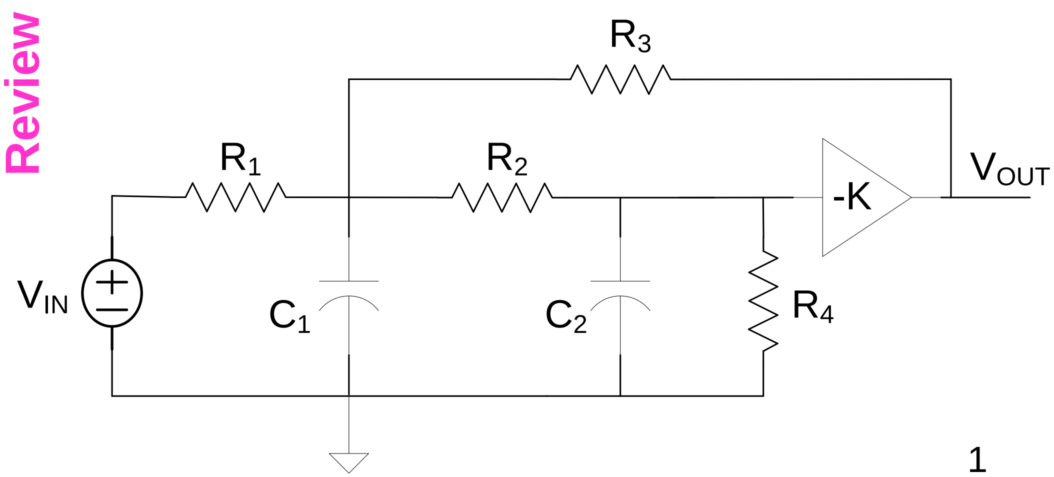
What causes the dramatic differences in performance between these two structures?
How can the performance of different structures be compared in general?

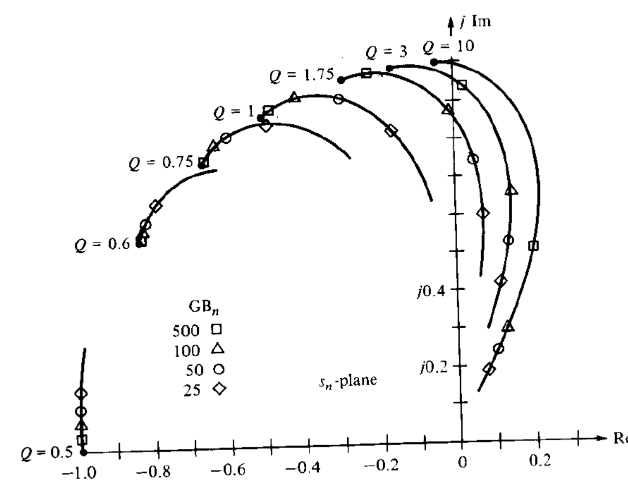


$$T(s)=K\frac{\dfrac{1}{R_1R_2C_1C_2}}{s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{1-K}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}}$$



Equal R, Equal C, Q=10 Pole Locus vs GB_N

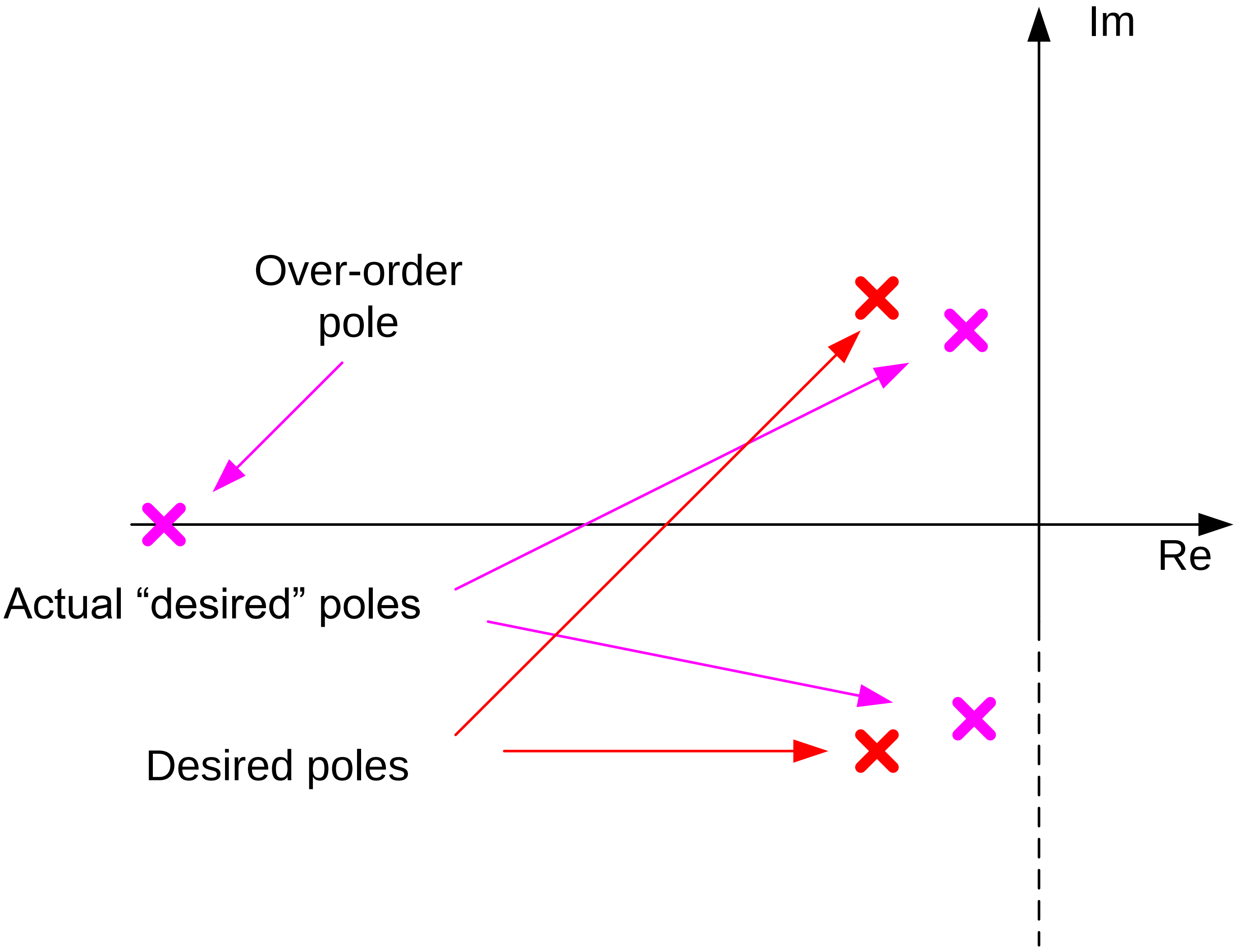




$$T(s)=-K\frac{\dfrac{1}{R_1R_2C_1C_2}}{s^2+s\left[\dfrac{1}{R_1C_1}\left(1+\dfrac{R_1}{R_3}\right)+\dfrac{1}{R_4C_2}+\dfrac{1}{R_2C_2}\left(1+\dfrac{C_2}{C_1}\right)\right]+\left[\dfrac{1+(R_1/R_3)(1+K)+(R_1/R_4)\left(1+(R_2/R_3)+(R_2/R_1)\right)}{R_1R_2C_1C_2}\right]}$$

Review from last time

Effects of GB on poles of KRC and -KRC Lowpass Filters



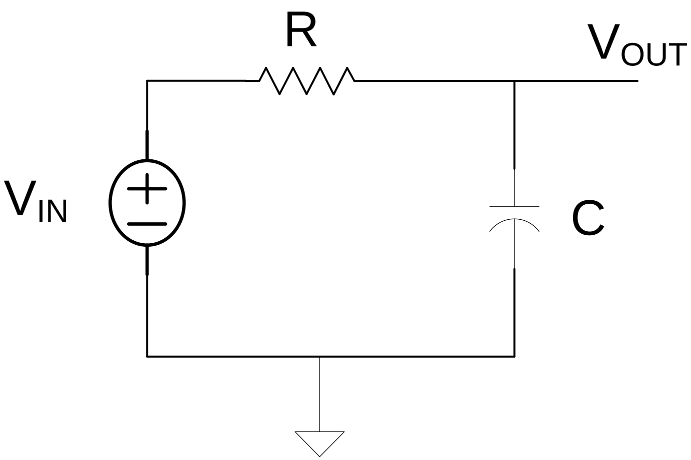
$$\frac{dF}{F} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

Dependent on circuit structure
(for some circuits, also not dependent on components)

Dependent only on components
(not circuit structure)

Review from last time

Consider:



$$T(s) = \frac{1}{1+RCs}$$

$$T(s) = \frac{\omega_0}{s + \omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

Review from last time

Theorem: If $f(x_1, ..x_m)$ can be expressed as $f = x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then $S_{x_i}^{f}$ is not dependent upon any of the variables in the set $\{x_1, ..x_m\}$

Proof:

$$S_{x_i}^{f} = S_{x_i}^{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \frac{\partial X_i^{\alpha_i}}{\partial x_i} \bullet \frac{x_i}{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \alpha_i X_i^{\alpha_i - 1} \bullet \frac{x_i}{X_i^{\alpha_i}}$$

$$S_{x_i}^{X_i^{\alpha_i}} = \alpha_i$$

$$S_{x_i}^{f} = \alpha_i$$

It is often the case that functions of interest are of the form expressed in the hypothesis of the theorem, and in these cases the previous claim is correct

Review from last time

Theorem: If $f(x_1, ..x_m)$ can be expressed as $f = x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then $S_{x_i}^f$ is not dependent upon any of the variables in the set $\{x_1, ..x_m\}$

Review from last time

Theorem: If $f(x_1, ..x_m)$ can be expressed as $f = x_1^{\alpha_1} x_2^{\alpha_2} ... x_m^{\alpha_m}$

where $\{\alpha_1, \alpha_2, \ldots \alpha_m\}$ are real numbers, then the sensitivity terms in

$$\frac{df}{f} = \sum_{i=1}^{k} \left(S_{x_i}^{f} \Big|_{\vec{X}_N} \bullet \frac{dx_i}{x_{iN}} \right)$$

are dependent only upon the circuit architecture and not dependent upon the components and and the right terms are dependent only upon the components and not dependent upon the architecture

This observation is useful for comparing the performance of two or more circuits where the function f shares this property

Metrics for Comparing Circuits

Summed Sensitivity

$$\rho_S = \sum_{i=1}^{m} S_{x_i}^{f}$$

Not very useful because sum can be small even when individual sensitivities are large

Schoeffler Sensitivity

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

Strictly heuristic but does differentiate circuits with low sensitivities from those with high sensitivities

Metrics for Comparing Circuits

$$\rho = \sum_{i=1}^{m} \left| S_{x_i}^{f} \right|$$

Often will consider several distinct sensitivity functions to consider effects of different components

$$\rho_R = \sum_{All\ resistors} \left| S_{R_i}^{f} \right|$$

$$\rho_C = \sum_{All\ capacitors} \left| S_{C_i}^{f} \right|$$

$$\rho_{OA} = \sum_{All\ op\ amps} \left| \mathcal{S}_{\tau_i}^{f} \right|$$

Homogeniety (defn)

A function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ if

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

Note: f may be comprised of more than n variables

Review from last time

Theorem: If a function f is homogeneous of order m in the n variables $\{x_1, x_2, \ldots x_n\}$ then

$$\sum_{i=1}^{n} S^{f}_{x_i} = m$$

$$f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^m f(x_1, x_2, \ldots x_n)$$

The concept of homogeneity and this theorem were somewhat late to appear

Are there really any useful applications of this rather odd observation?

Review from last time

Theorem: If all op amps in a filter are ideal, then ω_o, Q, BW, all band edges, and all poles and zeros are homogeneous of order 0 in the impedances.

Theorem: If all op amps in a filter are ideal and if T(s) is a dimensionless transfer function, T(s), T(jω), $|T(j\omega)|$, $\angle T\left(j\omega\right)$, are homogeneous of order 0 in the impedances

Review from last time

Theorem 1: If all op amps in a filter are ideal and if $T(s)$ is an impedance transfer function, $T(s)$ and $T(j\omega)$ are homogeneous of order 1 in the impedances

Theorem 2: If all op amps in a filter are ideal and if $T(s)$ is a conductance transfer function, $T(s)$ and $T(j\omega)$ are homogeneous of order -1 in the impedances

Corollary 1: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if a function f is homogeneous of order 0 in the impedances, then

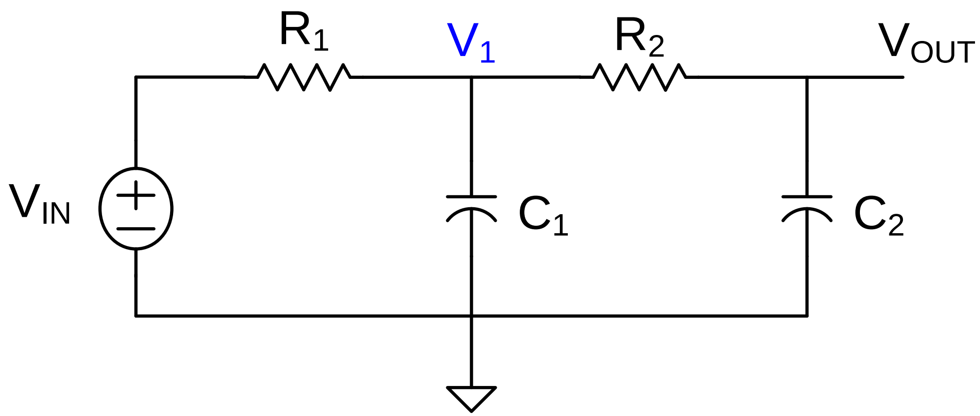
$$\sum_{i=1}^{k_1} S_{R_i}^{f} = \sum_{i=1}^{k_2} S_{C_i}^{f}$$

Corollary 2: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors then

$$\sum_{i=1}^{k_1} S_{R_i}^{Q} = 0$$

$$\sum_{i=1}^{k_2} S_{C_i}^{Q} = 0$$

Example



Determine the passive Q sensitivities $S_{R_1}^{Q}$ $S_{R_2}^{Q}$ $S_{C_1}^{Q}$ $S_{C_2}^{Q}$

$$\left.\begin{aligned} V_{OUT}\left(sC_1+G_2\right) &= V_1 G_2 \\ V_1\left(sC_1+G_1+G_2\right) &= V_{IN}G_1 + V_{OUT}G_2 \end{aligned}\right\}$$

$$T(s) = \frac{1}{s^2\left(R_1R_2C_1C_2\right)+s\left(R_1C_1+R_1C_2+R_2C_2\right)+1}$$

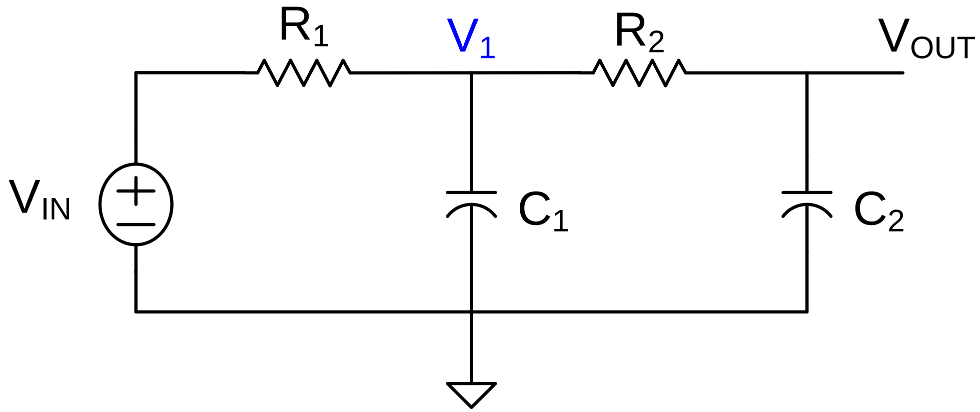
$$\omega_0 = \frac{1}{\sqrt{R_1R_2C_1C_2}}$$

$$Q = \frac{\sqrt{R_1R_2C_1C_2}}{R_1C_1+R_1C_2+R_2C_2}$$

By the definition of sensitivity, it follows that

$$S_{R_1}^{Q} = \frac{\left(R_1C_1+R_1C_2+R_2C_1\right)\frac{1}{2}\left(R_1R_2C_1C_2\right)^{-1/2}R_2C_1C_2-\left(C_1+C_2\right)\left(R_1R_2C_1C_2\right)^{1/2}}{\left(R_1C_1+R_1C_2+R_2C_2\right)^2} \bullet \frac{R_1}{Q}$$

Example



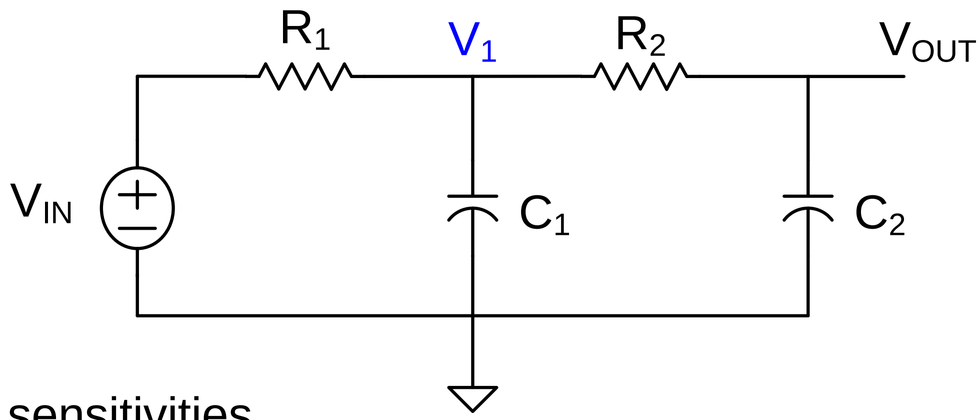
Determine the passive Q sensitivities $S_{R_1}^{Q}$ $S_{R_2}^{Q}$ $S_{C_1}^{Q}$ $S_{C_2}^{Q}$

$$S_{R_1}^{Q}=\frac{\left(R_1C_1+R_1C_2+R_2C_1\right)\frac{1}{2}\left(R_1R_2C_1C_2\right)^{-1/2}R_2C_1C_2-\left(C_1+C_2\right)\left(R_1R_2C_1C_2\right)^{1/2}}{\left(R_1C_1+R_1C_2+R_2C_2\right)^2}\bullet\frac{R_1}{Q}$$

Following some tedious manipulations, this simplifies to

$$S_{R_1}^{Q}=\frac{1}{2}-\frac{R_1\left(C_1+C_2\right)}{R_1C_1+R_1C_2+R_2C_2}$$

Example



Determine the passive Q sensitivities

Following the same type of calculations, can obtain

$$S_{R_1}^{Q}=\frac{1}{2}-\frac{R_1\left(C_1+C_2\right)}{R_1C_1+R_1C_2+R_2C_2} \qquad S_{R_2}^{Q}=\frac{1}{2}-\frac{R_2C_2}{R_1C_1+R_1C_2+R_2C_2}$$

$$S_{C_1}^{Q}=\frac{1}{2}-\frac{R_1C_1}{R_1C_1+R_1C_2+R_2C_2} \qquad S_{C_2}^{Q}=\frac{1}{2}-\frac{C_2\left(R_1+R_2\right)}{R_1C_1+R_1C_2+R_2C_2}$$

Verify

$$\sum_{i=1}^{k_2} S_{C_i}^{Q}=0 \qquad \sum_{i=1}^{k_1} S_{R_i}^{Q}=0$$

Could have saved considerable effort in calculations by using these theorems after $S_{R_1}^{Q}$ and $S_{C_1}^{Q}$ were calculated

Corollary 3: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if p_k is any pole and z_h is any zero, then

$$\sum_{i=1}^{k_1} S_{R_i}^{p_k} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{p_k} = -1$$

and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

Corollary 3: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if p_k is any pole and z_h is any zero, then

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and

$$\sum_{i=1}^{k_1} S_{R_i}^{z_h} = -1 \qquad \sum_{i=1}^{k_2} S_{C_i}^{z_h} = -1$$

Proof:

It was shown that scaling the frequency dependent elements by a factor η divides the pole (or zero) by η

Thus roots (poles and zeros) are homogeneous of order -1 in the frequency scaling elements

Proof:

Thus roots (poles and zeros) are homogeneous of order -1 in the frequency scaling elements

(For more generality, assume k_3 inductors)

$$\sum_{i=1}^{k_2} S_{C_i}^{p} + \sum_{i=1}^{k_3} S_{L_i}^{p} = -1 \qquad (1)$$

Since impedance scaling does not affects the poles, they are homogenous of order 0 in the impedances

$$\sum_{i=1}^{k_1} S_{R_i}^{p} + \sum_{i=1}^{k_2} S_{1/C_i}^{p} + \sum_{i=1}^{k_3} S_{L_i}^{p} = 0 \qquad (2)$$

Since there are no inductors in an active RC network, is follows from (1) that

$$\sum_{i=1}^{k_2} S_{C_i}^{p} = -1$$

And then from (2) and the theorem about sensitivity to reciprocals that

$$\sum_{i=1}^{k_1} S_{R_i}^{p} = -1$$

Corollary 4: If all op amps in an RC active filter are ideal and there are k_1 resistors and k_2 capacitors and if Z_{IN} is any input impedance of the network, then

$$\sum_{i=1}^{k_1} S_{R_i}^{Z_{IN}} - \sum_{i=1}^{k_2} S_{C_i}^{Z_{IN}} = 1$$

Claim: If op amps in the filters considered previously are not ideal but are modeled by a gain $A(s)=1/(\tau s)$, then all previous summed sensitivities developed for ideal op amps hold provided they are evaluated at the nominal value of $\tau=0$

Sensitivity Analysis

If a closed-form expression for a function f is obtained, a straightforward but tedious analysis can be used to obtain the sensitivity of the function to any components

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

Consider:

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

Closed-form expressions for T(s), T(jω), |T(jω)|, $\angle T(j\omega)$, a_i, b_i, can be readily obtained

Sensitivity Analysis

If a closed-form expression for a function f is obtained, a straightforward but tedious analysis can be used to obtain the sensitivity of the function to any components

$$S_x^f = \frac{\partial f}{\partial x} \bullet \frac{x}{f}$$

Consider:

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = K \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

Closed-form expressions for p_i, z_i, pole or zero Q, pole or zero ω_0, peak gain, ω_{3dB}, BW, … (generally the most critical and useful circuit characteristics) are difficult or impossible to obtain !

Bilinear Property of Electrical Networks

Theorem: Let x be any component or Op Amp time constant (1st order Op Amp model) of any linear active network employing a finite number of amplifiers and lumped passive components. Any transfer function of the network can be expressed in the form

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

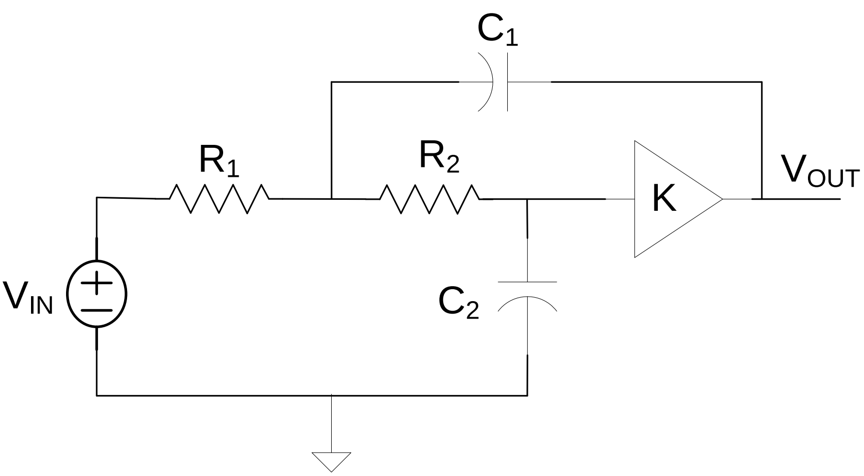
where N_0, N_1, D_0, and D_1 are polynomials in s that are not dependent upon x

A function that can be expressed as given above is said to be a bilinear function in the variable x and this is termed a bilateral property of electrical networks.

The bilinear relationship is useful for

1. Checking for possible errors in an analysis
2. Pole sensitivity analysis

Example of Bilinear Property : +KRC Lowpass Filter



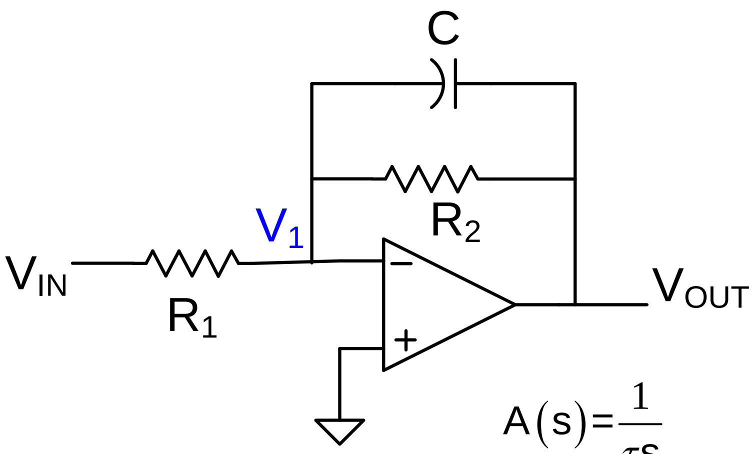
$$T(s)=\frac{\dfrac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{(1-K_0)}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}+K_0\tau s\left(s^2+s\left[\dfrac{1}{R_1C_1}+\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right]+\dfrac{1}{R_1R_2C_1C_2}\right)}$$

Consider R_1

$$T(s)=\frac{\dfrac{K_0}{R_2C_1C_2}}{R_1s^2+s\left[\dfrac{1}{C_1}+R_1\dfrac{1}{R_2C_1}+R_1\dfrac{(1-K_0)}{R_2C_2}\right]+\dfrac{1}{R_2C_1C_2}+K_0\tau s\left(R_1s^2+s\left[\dfrac{1}{C_1}+R_1\dfrac{1}{R_2C_1}+R_1\dfrac{1}{R_2C_2}\right]+\dfrac{1}{R_2C_1C_2}\right)}$$

$$T(s)=\frac{\left[\dfrac{K_0}{R_2C_1C_2}\right]+R_1\bullet[0]}{\left[s\dfrac{1}{C_1}+\dfrac{1}{R_2C_1C_2}+K_0\tau s\left(s\dfrac{1}{C_1}\right)+\dfrac{1}{R_2C_1C_2}\right]+R_1\left[s^2+s\left[\dfrac{1}{R_2C_1}+\dfrac{(1-K_0)}{R_2C_2}\right]+K_0\tau s\left(s^2+s\left[\dfrac{1}{R_2C_1}+\dfrac{1}{R_2C_2}\right]\right)\right]}$$

Example of Bilinear Property



$$A(s)=\frac{1}{\tau s}$$

$$\left.\begin{aligned} V_1\left(G_1+G_2+sC\right) &= V_{IN}G_1+V_{OUT}\left(sC+G_2\right) \\ V_{OUT} &= -V_1\left(\frac{1}{\tau s}\right) \end{aligned}\right\}$$

$$T(s) = \frac{-R_2}{R_1+R_1R_2Cs+\tau s\left(sCR_1R_2+R_1+R_2\right)}$$

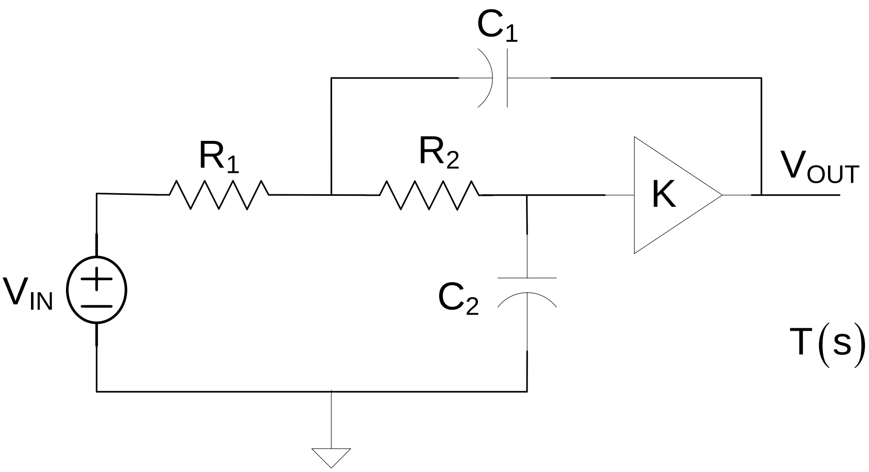
Consider R_1

$$T(s) = \frac{-R_2+0\bullet R_1}{\left[\tau sR_2\right]+R_1\left[1+R_2Cs+\tau s\left(sCR_2+1\right)\right]}$$

Consider τ

$$T(s) = \frac{-R_2+0\bullet \tau}{\left[R_1\left(1+R_2Cs\right)\right]+\tau\left[sR_2+sR_1\left(sCR_2+1\right)\right]}$$

Example of Bilinear Property : +KRC Lowpass Filter



Equal R Equal C

$$T(s)=\frac{\frac{K_0}{R^2C^2}}{s^2+s\left[\frac{(3-K_0)}{RC}\right]+\frac{1}{R^2C^2}+K_0\tau s\left(s^2+s\left[\frac{3}{RC}\right]+\frac{1}{R^2C^2}\right)}$$

$$T(s)=\frac{K_0}{R^2\left(C^2s^2+K_0\tau sC^2\right)+R\left(sC(3-K_0)+3K_0C\tau s^2\right)+(1+K_0\tau s)}$$

Can not eliminate the R^2 term

- Bilinear property only applies to individual components
- Bilinear property was established only for T(s)

Root Sensitivities

Consider expressing T(s) as a bilinear fraction in x

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}=\frac{N(s)}{D(s)}$$

Theorem: If z_i is any simple zero and/or p_i is any simple pole of T(s), then

$$S_x^{z_i}=\left(\frac{x}{z_i}\right)\left(\frac{-N_1(z_i)}{\frac{dN(z_i)}{dz_i}}\right) \quad \text{and} \quad S_x^{p_i}=\left(\frac{x}{p_i}\right)\left(\frac{-D_1(p_i)}{\frac{dD(p_i)}{dp_i}}\right)$$

Note: Do not need to find expressions for the poles or the zeros to find the pole and zero sensitivities !

Note: Do need the poles or zeros but they will generally be known by design

Note: Will make minor modifications for extreme values for x (i.e. τ for op amps)

Root Sensitivities

Theorem: If p_i is any simple pole of T(s), then

$$S_x^{p_i} = \left(\frac{x}{p_i}\right)\left(\frac{-D_1(p_i)}{\frac{dD(p_i)}{dp_i}}\right)$$

Proof (similar argument for the zeros)

$$D(s)=D_0(s)+xD_1(s)$$

By definition of a pole,

$$D(p_i)=0$$

$$\therefore \quad D(p_i)=D_0(p_i)+xD_1(p_i)=0$$

Root Sensitivities

$$\therefore \qquad D(p_i) = D_0(p_i) + xD_1(p_i)$$

Differentiating this expression implicitly WRT x, we obtain

$$\frac{\partial D_0(p_i)}{\partial p_i}\frac{\partial p_i}{\partial x} + \left[x\frac{\partial D_1(p_i)}{\partial p_i}\frac{\partial p_i}{\partial x} + D_1(p_i)\right] = 0$$

Re-grouping, obtain

$$\frac{\partial p_i}{\partial x}\left[\frac{\partial D_0(p_i)}{\partial p_i} + x\frac{\partial D_1(p_i)}{\partial p_i}\right] = -D_1(p_i)$$

But term in brackets is derivative of $D(p_i)$ wrt p_i, thus

$$\frac{\partial p_i}{\partial x} = -\frac{D_1(p_i)}{\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)}$$

Root Sensitivities

$$\frac{\partial p_i}{\partial x} = -\frac{D_1(p_i)}{\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)}$$

Finally, from the definition of sensitivity,

$$S_x^{p_i} = \frac{x}{p_i}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{p_i}\right)\frac{D_1(p_i)}{\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)}$$

Root Sensitivities

$$S_x^{p_i} = \frac{x}{p_i}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{p_i}\right)\frac{D_1(p_i)}{\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)}$$

Observation: Although the sensitivity expression is readily obtainable, direction information about the pole movement is obscured because the derivative is multiplied by the quantity p_i which is often complex. Usually will use either

$$\mathcal{S}_x^{p_i} = \frac{\partial p_i}{\partial x}$$

or

$$\tilde{S}_x^{p_i} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x} = -\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)}$$

which preserve direction information when working with pole or zero sensitivity analysis.

Root Sensitivities

Summary: Pole (or zero) locations due to component variations can be approximated with simple analytical calculations without obtaining parametric expressions for the poles (or zeros).

$$p_i \simeq p_i\Big|_{\substack{\text{Ideal}\\ \text{Components}}} + \Delta p_i$$

where

$$\Delta p_i \simeq \Delta x \bullet \mathcal{S}_x^{p_i}$$

$$\mathcal{S}_x^{p_i} = -\frac{D_1(p_i)}{\left.\left(\dfrac{\partial D(p_i)}{\partial p_i}\right)\right|_{p_{iN}}}$$

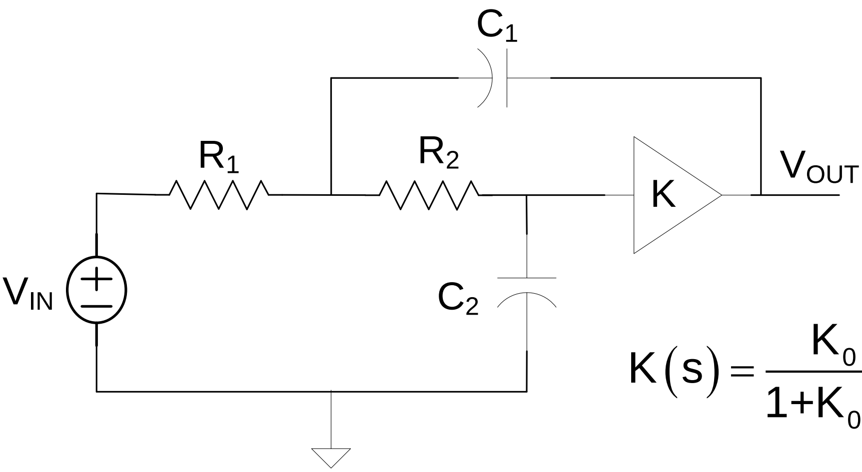
and

$$D(s) = D_0(s) + x \bullet D_1(s)$$

Alternately,

$$\Delta p_i \simeq \left(|p_i|\frac{\Delta x}{x}\right)\tilde{S}_x^{p_i}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

$$K(s)=\frac{K_0}{1+K_0\tau s}$$

$$T(s)=\frac{\frac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}+K_0\tau s\left(s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}\right)}$$

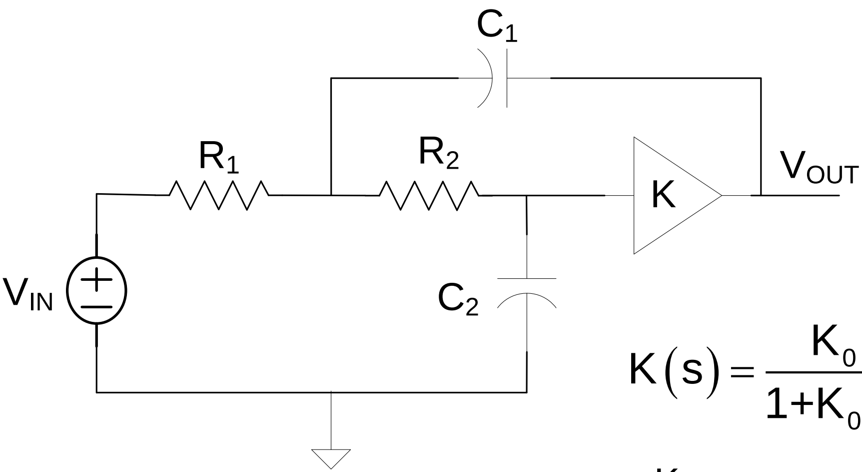
write in bilinear form

$$T(s)=\frac{\frac{K_0}{R_2C_1C_2}}{\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+K_0\tau s\left(s\frac{1}{C_1}\right)+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}+K_0\tau s\left(s^2+s\left[\frac{1}{R_2C_1}+\frac{1}{R_2C_2}\right]\right)\right]\right]}$$

evaluate at τ=0

$$T(s)=\frac{\frac{K_0}{R_2C_1C_2}}{\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]\right]}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

$$K(s)=\frac{K_0}{1+K_0\tau s}$$

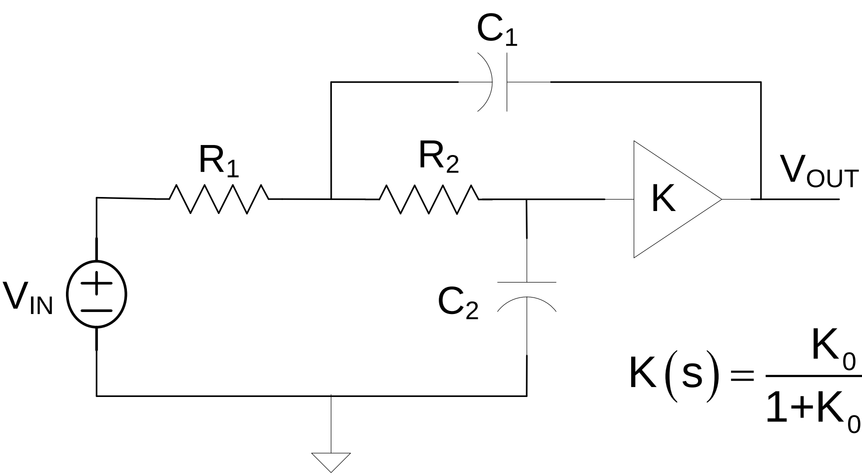
$$T(s)=\frac{\frac{K_0}{R_2C_1C_2}}{\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]\right]}$$

$$D_1(s)=s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]$$

$$D(s)=\left(s\frac{1}{C_1}+\frac{1}{R_2C_1C_2}+\frac{1}{R_2C_1C_2}\right)+R_1\left[s^2+s\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]\right]=R_1\left(s^2+s\left[\frac{\omega_0}{Q}\right]+\omega_0^2\right)$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{1}{|p_i|}\right)\frac{p^2+p\left[\frac{1}{R_2C_1}+\frac{(1-K_0)}{R_2C_2}\right]}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$K(s)=\frac{K_0}{1+K_0\tau s}$$

$$T(s)=\frac{N_0(s)+xN_1(s)}{D_0(s)+xD_1(s)}$$

T

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{x}{|p_i|}\right)\frac{D_1(p_i)}{\left(\frac{\partial D(p_i)}{\partial p_i}\right)}$$

$$T(s)=\frac{\frac{K_0}{R_1R_2C_1C_2}}{s^2+s\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}}$$

$$p^2+p\left[\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}\right]+\frac{1}{R_1R_2C_1C_2}=0$$

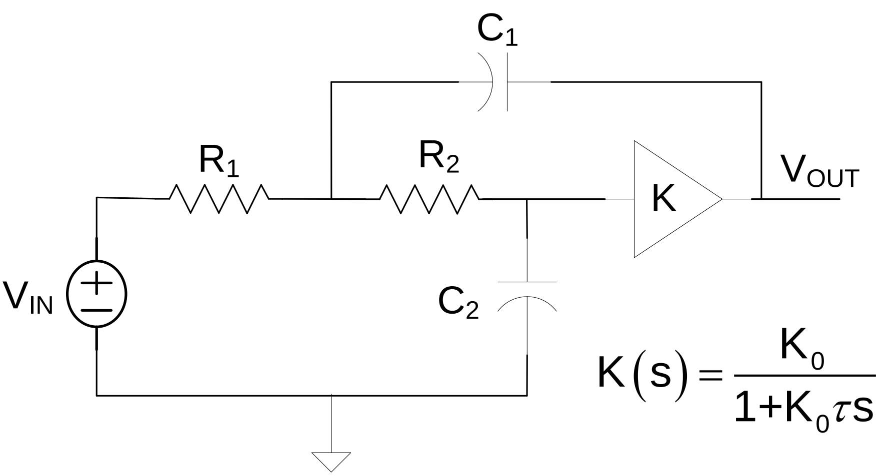
$$p^2+p\left[\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}\right]=-\frac{1}{R_1R_2C_1C_2}-p\frac{1}{R_1C_1}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=-\left(\frac{1}{|p_i|}\right)\frac{p^2+p\left[\frac{1}{R_2C_1}+\frac{(1\text{-}K_0)}{R_2C_2}\right]}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\left(\frac{1}{|p_i|}\right)\frac{\frac{1}{R_1R_2C_1C_2}+p\frac{1}{R_1C_1}}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}=\left(\frac{1}{\omega_0}\right)\frac{\omega_0^2+p\frac{1}{R_1C_1}}{\left(2p_i+\frac{\omega_0}{Q}\right)}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$K(s) = \frac{K_0}{1+K_0\tau s}$$

$$\tilde{S}_x^{p_i} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x} = \left(\frac{1}{\omega_0}\right)\frac{\omega_0^2 + p\frac{1}{R_1C_1}}{\left(2p_i + \frac{\omega_0}{Q}\right)}$$

For equal R, equal C $\quad \omega_0 = \frac{1}{RC}$

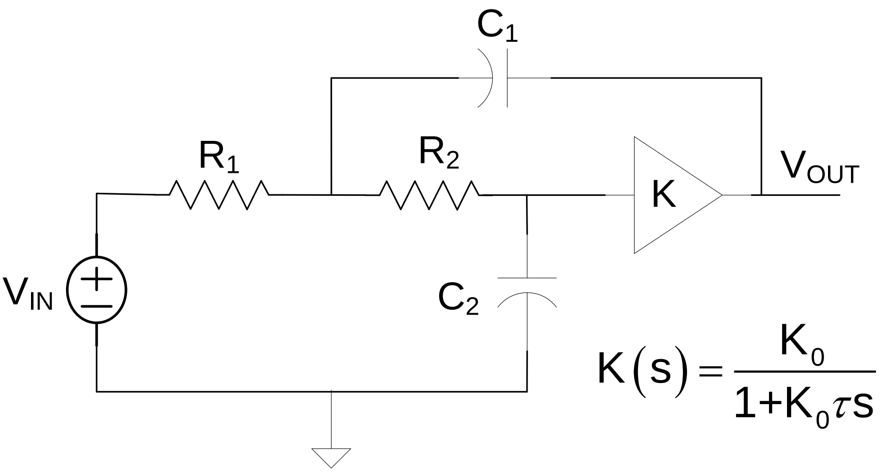
$$\tilde{S}_{R_1}^{p} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x} = \left(\frac{1}{\omega_0}\right)\frac{\omega_0^2 + p\omega_0}{\left(2p_i + \frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p} = \frac{\omega_0 - \frac{\omega_0}{2Q} \pm \frac{\omega_0}{2Q}\sqrt{1\text{-}4Q^2}}{\pm\frac{\omega_0}{Q}\sqrt{1\text{-}4Q^2}}$$

$$\tilde{S}_{R_1}^{p} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x} = \frac{\omega_0 + p}{\left(2p + \frac{\omega_0}{Q}\right)}$$

$$\tilde{S}_{R_1}^{p} = \frac{Q\text{-}\frac{1}{2} \pm \frac{1}{2}\sqrt{1\text{-}4Q^2}}{\pm\sqrt{1\text{-}4Q^2}}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$K(s)=\frac{K_0}{1+K_0\tau s}$$

$$\tilde{S}_x^{p_i}=\frac{x}{|p_i|}\frac{\partial p_i}{\partial x}$$

For equal R, equal C

$$\tilde{S}_{R_1}^{p}=\frac{Q-\frac{1}{2}\pm\frac{1}{2}\sqrt{1\text{-}4Q^2}}{\pm\sqrt{1\text{-}4Q^2}}$$

Note this contains magnitude and direction information

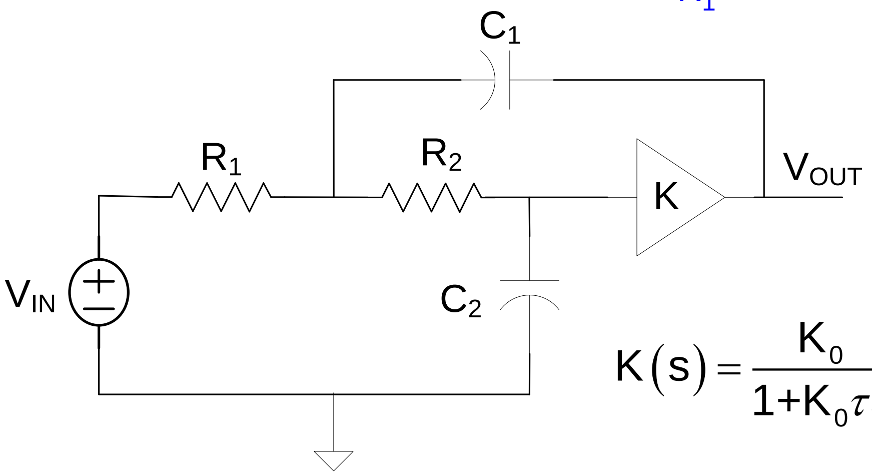
For high Q

$$\tilde{S}_{R_1}^{p}=\frac{Q\pm\frac{1}{2}\sqrt{\text{-}4Q^2}}{\pm\sqrt{\text{-}4Q^2}}=\frac{Q\pm jQ}{\pm j2Q}=\frac{1\pm j}{\pm j2}=\frac{j\pm 1}{\pm 2}=\frac{1}{2}\pm\frac{1}{2}j$$

$$\Delta p_i\cong|p_i|\tilde{S}_x^{p_i}\frac{\Delta x}{x}$$

$$\Delta p_i\cong\omega_0(0.5\pm 0.5j)\frac{\Delta R_1}{R_1}$$

Example: Determine $\tilde{S}_{R_1}^{p_i}$ for the +KRC Lowpass Filter for equal R, equal C



$$\tilde{S}_x^{p_i} = \frac{x}{|p_i|}\frac{\partial p_i}{\partial x}$$

For equal R, equal C

$$K(s) = \frac{K_0}{1+K_0\tau s}$$

For high Q

$$\Delta p_i \cong \omega_0 (0.5 \pm 0.5j) \frac{\Delta R_1}{R_1}$$

Could we have assumed equal R equal C before calculation?

No ! Analysis would not apply (not bilinear)

Results would obscure effects of variations in individual components

Was this a lot of work for such a simple result?

Yes ! But it is parametric and still only took maybe 20 minutes

But it needs to be done only once for this structure

Can do for each of the elements

What is the value of this result?

Understand how components affect performance of this circuit

Compare performance of different circuits for architecture selection

Transfer Function Sensitivities

$$S_x^{T(s)}\Big|_{s=j\omega} = S_x^{T(j\omega)}$$

$$S_x^{T(j\omega)} = S_x^{|T(j\omega)|} + j\theta S_x^{\theta}$$

where $\theta = \angle T(j\omega)$

$$S_x^{|T(j\omega)|} = \mathrm{Re}\left(S_x^{T(j\omega)}\right)$$

$$S_x^{\theta} = \frac{1}{\theta}\mathrm{Im}\left(S_x^{T(j\omega)}\right)$$

Transfer Function Sensitivities

If T(s) is expressed as $T(s) = \dfrac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = \dfrac{N(s)}{D(s)}$

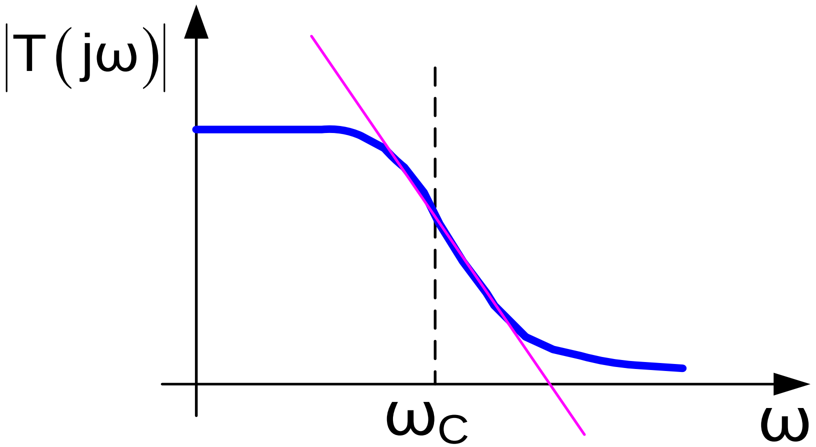
then
$$S_x^{T(s)} = \frac{\sum_{i=0}^{m} a_i s^i S_x^{a_i}}{N(s)} - \frac{\sum_{i=0}^{n} b_i s^i S_x^{b_i}}{D(s)}$$

If T(s) is expressed as $T(s) = \dfrac{N_0(s) + xN_1(s)}{D_0(s) + xD_1(s)}$

$$S_x^{T(s)} = \frac{x\left[D_0(s)N_1(s) - N_0(s)D_1(s)\right]}{\left(N_0(s) + xN_1(s)\right)\left(D_0(s) + xD_1(s)\right)}$$

Band-edge Sensitivities

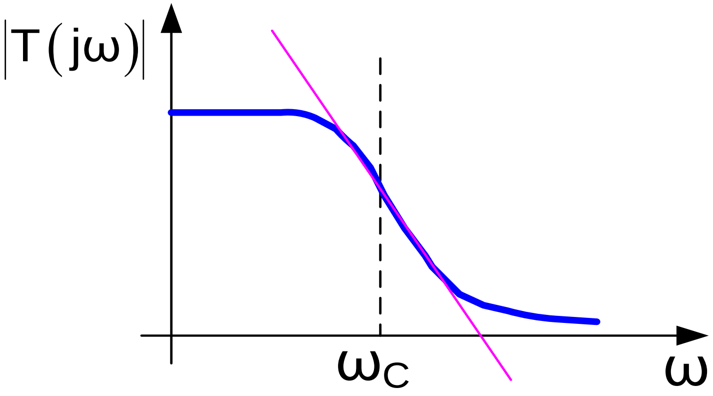
The band edge of a filter is often of interest. A closed-form expression for the band-edge of a filter may not be attainable and often the band-edges are distinct from the ω_0 of the poles. But the sensitivity of the band-edges to a parameter x is often of interest.



Want

$$S_x^{\omega_C} = \frac{\partial \omega_C}{\partial x} \bullet \frac{x}{\omega_C}$$

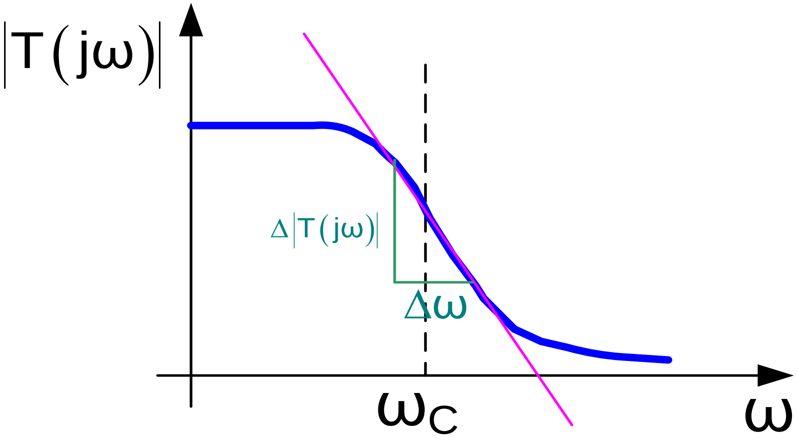
Band-edge Sensitivities



Theorem: The sensitivity of the band-edge of a filter is given by the expression

$$S_x^{\omega_C} = \frac{\left. S_x^{|T(j\omega)|} \right|_{\omega=\omega_C}}{\left. S_\omega^{|T(j\omega)|} \right|_{\omega=\omega_C}}$$

Band-edge Sensitivities

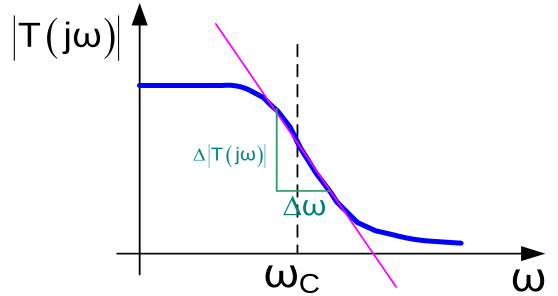


Proof:

Observe
$$\frac{\partial|T(j\omega)|}{\partial\omega} \cong \frac{\Delta|T(j\omega)|}{\Delta\omega}$$

$$\frac{\partial|T(j\omega)|}{\partial\omega} \cong \frac{\Delta|T(j\omega)|}{\Delta x} \bullet \frac{\Delta x}{\Delta\omega} \cong \frac{\dfrac{\partial|T(j\omega)|}{\partial x}}{\dfrac{\partial\omega}{\partial x}}$$

Band-edge Sensitivities



$$\frac{\partial |T(j\omega)|}{\partial \omega} \cong \frac{\Delta |T(j\omega)|}{\Delta x} \bullet \frac{\Delta x}{\Delta \omega} \cong \frac{\dfrac{\partial |T(j\omega)|}{\partial x}}{\dfrac{\partial \omega}{\partial x}}$$

$$\frac{\partial \omega}{\partial x} \cong \frac{\dfrac{\partial |T(j\omega)|}{\partial x}}{\dfrac{\partial |T(j\omega)|}{\partial \omega}}$$

$$\frac{\partial \omega}{\partial x} \cong \frac{\dfrac{\partial |T(j\omega)|}{\partial x} \bullet \dfrac{x}{|T(j\omega)|}}{\dfrac{\partial |T(j\omega)|}{\partial \omega} \bullet \dfrac{\omega}{|T(j\omega)|}} \left(\frac{\omega}{x}\right)$$

$$\frac{\partial \omega}{\partial x} \bullet \left(\frac{x}{\omega}\right) \cong \frac{\dfrac{\partial |T(j\omega)|}{\partial x} \bullet \dfrac{x}{|T(j\omega)|}}{\dfrac{\partial |T(j\omega)|}{\partial \omega} \bullet \dfrac{\omega}{|T(j\omega)|}}$$

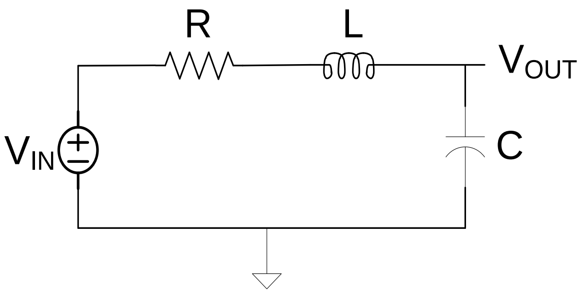
$$S_x^{\omega} = \frac{S_x^{|T(j\omega)|}}{S_\omega^{|T(j\omega)|}}$$

$$S_x^{\omega_C} = \frac{S_x^{|T(j\omega)|}\Big|_{\omega=\omega_C}}{S_\omega^{|T(j\omega)|}\Big|_{\omega=\omega_C}}$$

Sensitivity Comparisons

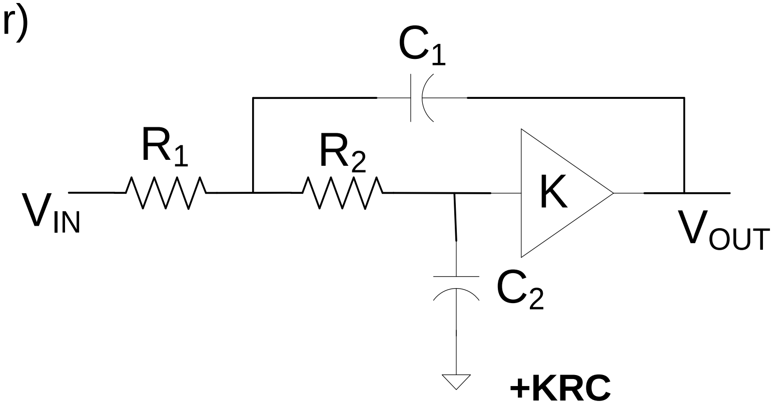
Consider 5 second-order lowpass filters

(all can realize same T(s) within a gain factor)



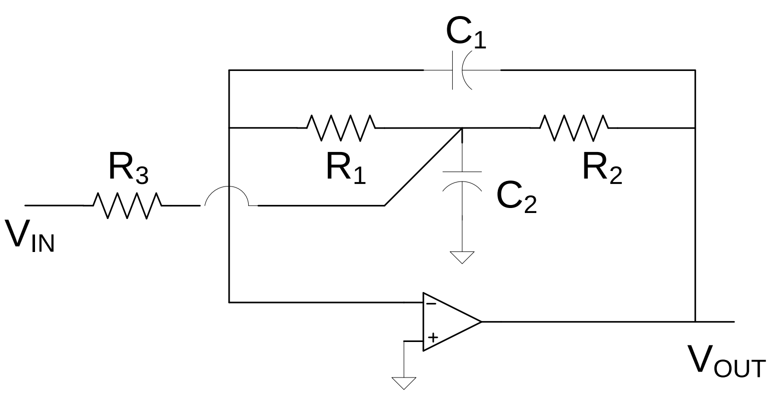
Passive RLC

(a)



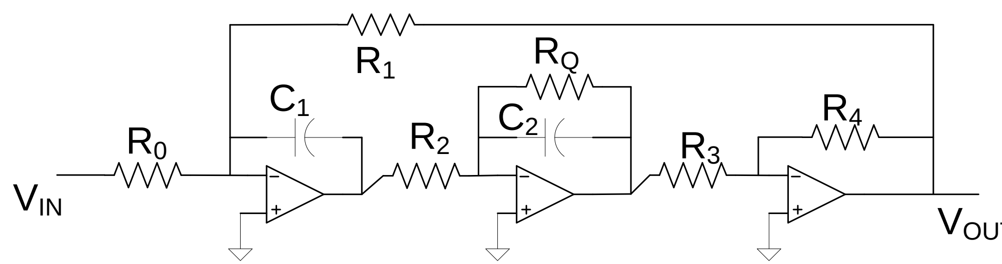
+KRC

(b)



Bridged-T Feedback

(c)



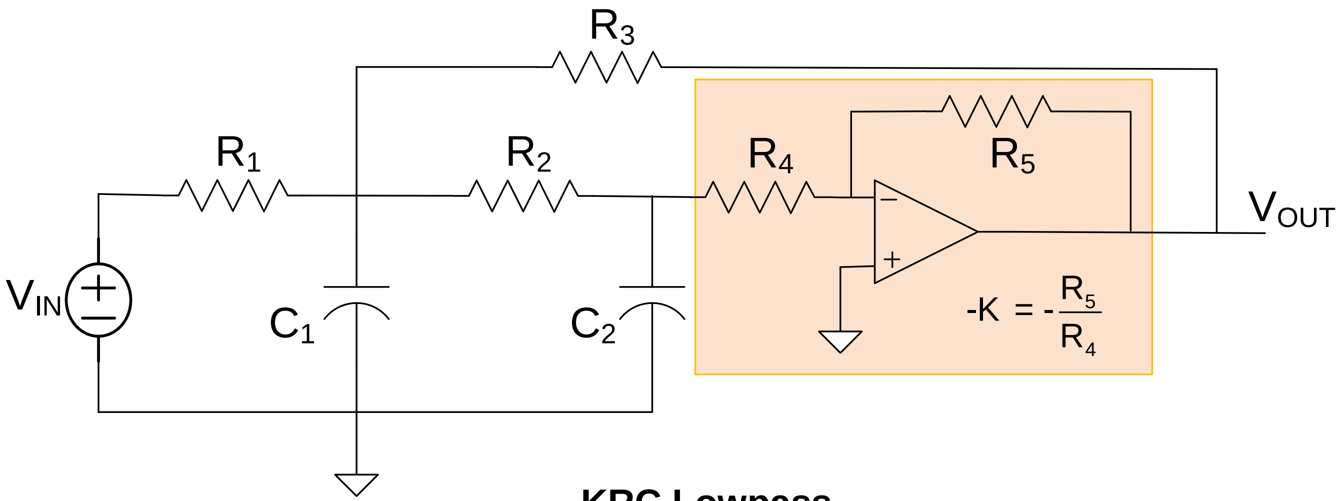
Two-Integrator Loop

(d)

Sensitivity Comparisons

Consider 5 second-order lowpass filters

(all can realize same T(s) within a gain factor)



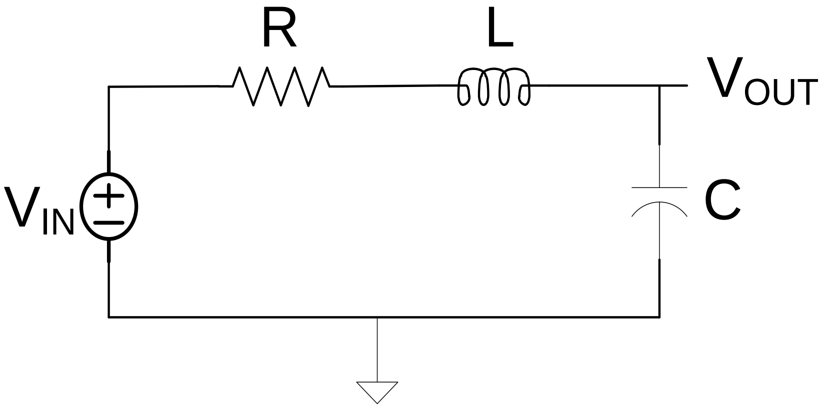
-KRC Lowpass

(e)

For all 5 structures, will have same transfer function within a gain factor

$$T(s) = \frac{K\omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

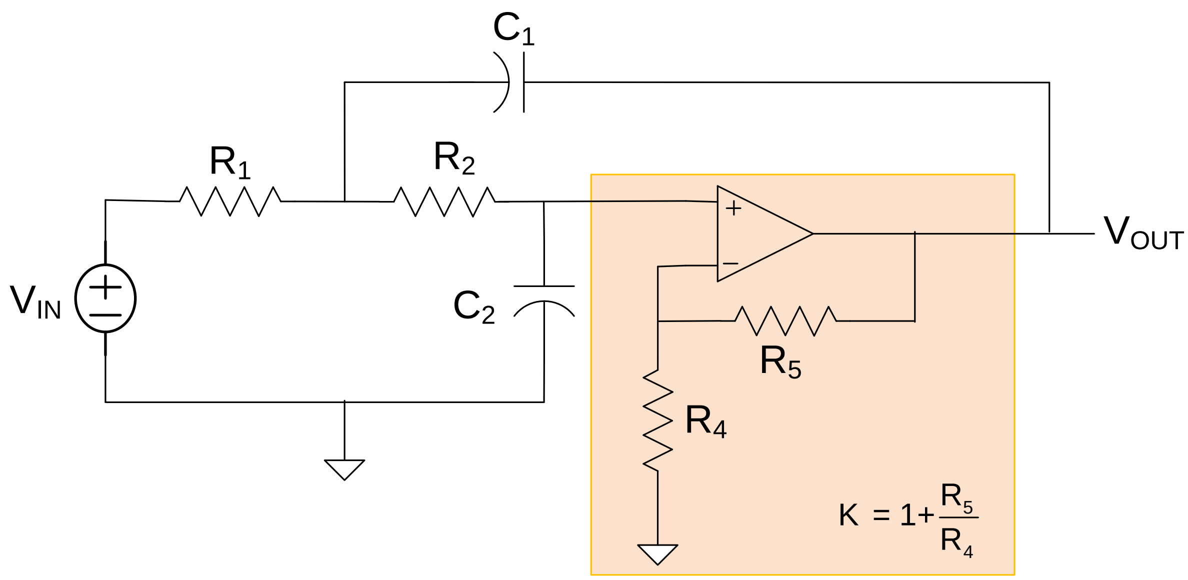
a) – Passive RLC



$$T(s) = \frac{V_{OUT}}{V_{IN}} = \frac{1/LC}{s^2 + s\frac{R}{L} + 1/LC}$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

b) + KRC (a Sallen and Key filter)



$$T(s) = \frac{\dfrac{K}{R_1R_2C_1C_2}}{s^2 + s\left[\left(\dfrac{1}{\sqrt{R_1R_2C_1C_2}}\right)\left(\sqrt{\dfrac{R_1C_1}{R_2C_2}} + \sqrt{\dfrac{R_2C_2}{R_1C_1}} + \sqrt{\dfrac{R_1C_2}{R_2C_1}} - K\sqrt{\dfrac{R_1C_1}{R_2C_2}}\right)\right] + \dfrac{1}{R_1R_2C_1C_2}}$$

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}}$$

$$Q = \frac{1}{\left(\sqrt{\dfrac{R_1C_1}{R_2C_2}} + \sqrt{\dfrac{R_2C_2}{R_1C_1}} + \sqrt{\dfrac{R_1C_2}{R_2C_1}} - K\sqrt{\dfrac{R_1C_1}{R_2C_2}}\right)}$$

Case b1 : Equal R, Equal C

$$R_1 = R_2 = R \qquad C_1 = C_2 = C$$

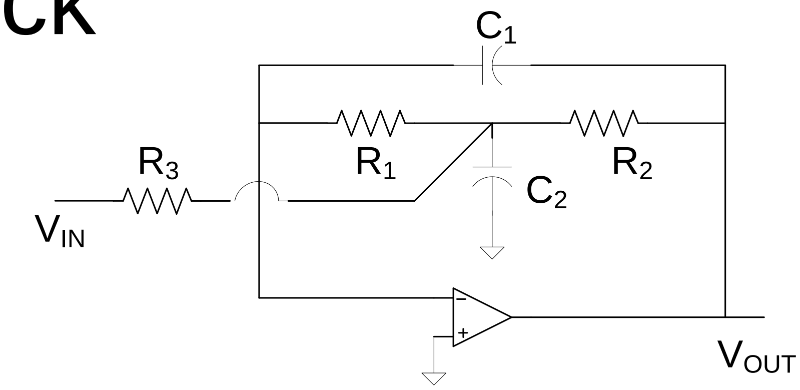
$$\omega_0 = \frac{1}{RC} \qquad K = 3 - \frac{1}{Q}$$

Case b2 : Equal R, K=1

$$R_1 = R_2 = R \qquad Q = \frac{1}{2}\sqrt{\frac{C_1}{C_2}}$$

$$T(s) = \frac{K\omega_0^2}{s^2 + s\frac{\omega_0}{Q} + \omega_0^2}$$

c) Bridged T Feedback



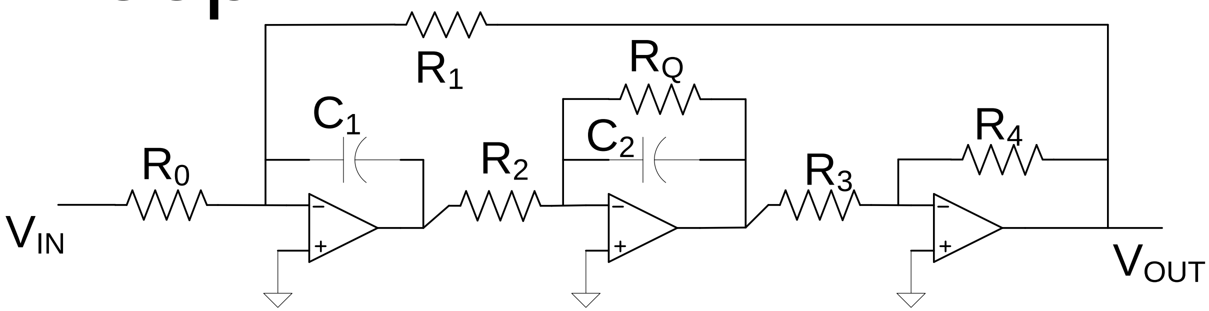
$$T(s) = \frac{\dfrac{1}{R_1R_3C_1C_2}}{s^2 + s\left[\left(\sqrt{\dfrac{C_2}{C_1}}\right)\left(\dfrac{1}{\sqrt{R_1R_2C_1C_2}}\right)\left(\sqrt{\dfrac{R_1}{R_3}} + \sqrt{\dfrac{R_2}{R_1}} + \dfrac{\sqrt{R_1R_2}}{R_3}\right)\right] + \dfrac{1}{R_1R_2C_1C_2}}$$

$$\omega_0 = \sqrt{\frac{1}{R_1R_2C_1C_2}} \qquad Q = \frac{1}{\left(\sqrt{\dfrac{C_2}{C_1}}\right)\left(\sqrt{\dfrac{R_1}{R_3}} + \sqrt{\dfrac{R_2}{R_1}} + \dfrac{\sqrt{R_1R_2}}{R_3}\right)}$$

If $R_1=R_2=R_3=R$ and $C_2=9Q^2C_1$

$$T(s) = \frac{\dfrac{1}{9Q^2R^2C_1^2}}{s^2 + s\left[\left(\dfrac{1}{3Q^2RC_1}\right)\right] + \dfrac{1}{9Q^2R^2C_1^2}}$$

d) 2 integrator loop



$$T(s) = -\frac{\frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}{s^2 + s\left(\frac{1}{R_Q C_2}\right) + \frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}$$

$$\omega_0 = \sqrt{\frac{R_4}{R_3} \bullet \frac{1}{R_0 R_2 C_1 C_2}}$$

$$Q = \frac{R_Q}{\sqrt{R_0 R_2}} \sqrt{\frac{C_2}{C_1}}$$

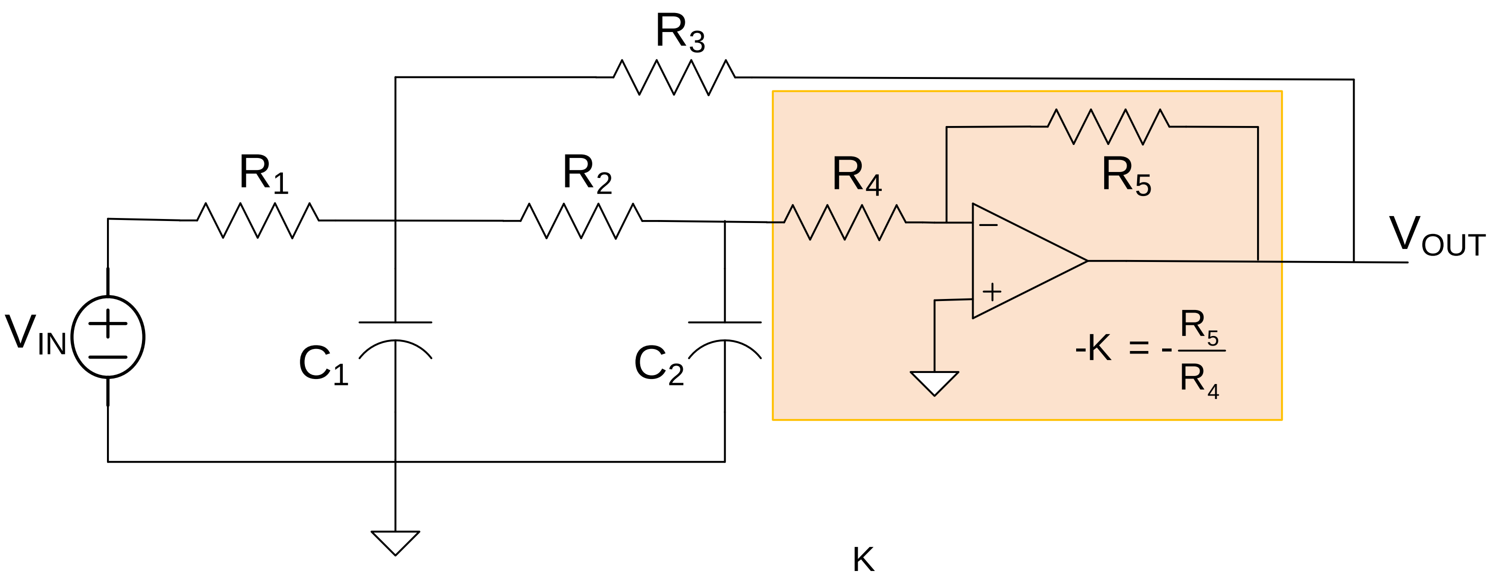
For: $R_0 = R_1 = R_2 = R$ $C_1 = C_2 = C$ $R_3 = R_4$

$$T(s) = -\frac{\frac{1}{R^2 C^2}}{s^2 + s\left(\frac{1}{R_Q C}\right) + \frac{1}{R^2 C^2}}$$

$$R_Q = QR$$

$$\omega_0 = \frac{1}{RC}$$

d) - KRC (a Sallen and Key filter)



$$T(s) = -\frac{\dfrac{K}{R_1R_2C_1C_2}}{s^2 + s\left[\left(1+\dfrac{R_1}{R_3}\right)\left(\dfrac{1}{R_1C_1}\right) + \left(1+\dfrac{C_2}{C_1}\right)\left(\dfrac{1}{R_2C_2}\right) + \left(\dfrac{1}{R_4C_2}\right)\right] + \dfrac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}$$

$$\omega_0 = \sqrt{\frac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}$$

$$Q = \frac{\sqrt{\dfrac{1+(R_1/R_3)(1+K)+(R_1/R_4)(1+R_2/R_3+R_2/R_1)}{R_1R_2C_1C_2}}}{\left(1+\dfrac{R_1}{R_3}\right)\left(\dfrac{1}{R_1C_1}\right) + \left(1+\dfrac{C_2}{C_1}\right)\left(\dfrac{1}{R_2C_2}\right) + \left(\dfrac{1}{R_4C_2}\right)}$$

Often $R_1=R_2=R_3=R_4=R$, $C_1=C_2=C$

$$Q = \frac{\sqrt{5+K_0}}{5}$$

Stay Safe and Stay Healthy !

End of Lecture 20